SOME CHARACTERIZATIONS OF THE SPHERICAL HARMONICS COEFFICIENTS FOR ISOTROPIC RANDOM FIELDS

Paolo Baldi

Dipartimento di Matematica

Università di Roma - Tor Vergata

Via della Ricerca Scientifica

00133 Roma, Italy.

baldi@mat.uniroma2.it

Domenico Marinucci

Dipartimento di Matematica

Università di Roma - Tor Vergata

Via della Ricerca Scientifica

00133 Roma, Italy.

marinucc@mat.uniroma2.it

Abstract

In this paper we provide some simple characterizations for the spherical harmonics coefficients of an isotropic random field on the sphere. The main result is a characterization of isotropic gaussian fields through independence of the coefficients of their development in spherical harmonics.

1 Introduction

In recent years the study of real random fields on the sphere has received much attention, as this topic is a necessary tool in the statistical study of the CMB (Cosmic Microwave Background) radiation. The existing physical literature is huge, with particular emphasis on testing for Gaussianity and isotropy (e.g., for a small sample, [5], [4], [11]) as both these issues have deep implications for cosmological physics. See for reviews on this subject [2] and [9], where many more references can be found. We also mention [6], [1] and [10] for a mathematical treatment of the subject.

A natural tool for this kind of enquiry is the development of the random field in a series of spherical harmonics. This raises some simple, but not so obvious questions. For an isotropic random field T, given its development in spherical harmonics (see §2),

$$T(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x) \tag{1}$$

what properties must be expected to be satisfied by the coefficient $a_{\ell m}$? Or, from a different point of view, which conditions must be verified by the coefficients in order that the development above defines an isotropic random field?

These questions are of interest both from a statistical point of view (for instance in order to devise a test suitable to detect non-Gaussianity or anisotropy) or from a probabilistic point of view (how to sample an isotropic random field).

In this paper we provide some results in this perspective. More precisely, first we prove that, for an isotropic random field, the coefficients are necessarily uncorrelated. This fact is well known, but we give a proof that holds without the assumption that the field is mean square continuous.

Then it is proved that each of the complex r.v.'s $a_{\ell m}$ has a distribution whose phase is uniform on $[-\pi, \pi]$. This implies in particular that, for any isotropic field, the ratio $\operatorname{Re} a_{\ell m} / \operatorname{Im} a_{\ell m}$ is necessarily distributed accordingly to a Cauchy distribution.

Finally, having remarked that if the field T is gaussian then the $a_{\ell m}$'s, $\ell = 0, 1, \ldots, m = 0, \ldots, \ell$ are independent, we prove that also the converse is true. Thus the only isotropic fields such that the $a_{\ell m}$'s, $\ell = 0, 1, \ldots, m = 0, \ldots, \ell$ are independent are those that are gaussian. This, which is the main result of this paper, is not a consequence of the central limit theorem, but results from some

classical characterization of gaussian random variables, through independence of some linear statistics.

This result gives a rigorous proof of claims that can be occasionally found in the cosmological literature (see [3] e.g.)

2 Isotropic random fields

In this section we recall some well known facts about isotropic random fields T defined on the unit sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. We assume that these fields are isotropic in the strong sense, that is, their probability law is invariant with respect to the action of $SO(3)$. The isotropy assumption can then be stated as follows: for all $g \in SO(3)$ and $x_1, \ldots, x_p \in S^2$, the two vectors

$$(T(gx_1), \ldots, T(gx_p)) \quad \text{and} \quad (T(x_1), \ldots, T(x_p)) \ ,$$

have the same distribution. Throughout this paper by "isotropic" we mean isotropic in this sense.

We shall use the spherical coordinates on S^2 i.e. $x = (\vartheta, \varphi)$, where $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$. Also we shall use for $g \in SO(3)$ the parameterization through the Euler angles $0 \leq \alpha, \gamma < 2\pi$ and $0 \leq \beta \leq \pi$. Assuming the right-hand side exists, for each $\ell = 1, 2, \ldots$ we can define the random vector

$$a_{\ell.} = \int_{S^2} T(x) Y_{\ell.}(x) \, dx \tag{2}$$

where $dx = \sin \vartheta \, d\varphi \, d\vartheta$ denotes the Lebesgue measure on S^2 and $Y_{\ell.}$ the vector of spherical harmonics defined by

$$Y_{\ell.}(\theta, \varphi) = (Y_{\ell\ell}(\theta, \varphi), \ldots, Y_{\ell,-\ell}(\theta, \varphi))' \ ,$$

$$Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell m}(\cos \theta) e^{im\varphi} \ , \text{ for } m \geq 0 \ ,$$

$$Y_{\ell m}(\theta, \varphi) = (-1)^m \overline{Y_{\ell,-m}(\theta, \varphi)} \ , \text{ for } m < 0 \ ; \tag{3}$$

here $P_{\ell m}(\cos \theta)$ denotes the associated Legendre functions i.e.

$$P_{\ell m}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \ , \ P_\ell(x) = \frac{1}{2^\ell l!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell,$$
$$m = 0, 1, 2, \ldots, \ell \ , \ \ell = 1, 2, 3, \ldots \ .$$

3

A detailed discussion of the properties of the spherical harmonics can be found in Varshalovich, Moskalev and Khersonskii [12], chapter 5. Our purpose in this paper is to provide some characterizations of the probability law of the vector $a_{\ell.}$ under the isotropy assumption. It is a standard result that the functions $Y_{\ell m}$ $m = -\ell, \ldots, \ell$ form a basis for the vector space of functions on S^2 which are restrictions of homogeneous harmonic polynomials of degree ℓ. This vector space being invariant by the action of $SO(3)$, for any ℓ and $g \in SO(3)$ there exist a $(2\ell + 1) \times (2\ell + 1)$ matrix $D^\ell(g)$ such that

$$Y_{\ell.}(gx) = D^\ell(g)Y_\ell(x) , \tag{4}$$

This provides a representation of $SO(3)$ on $\mathbb{C}^{2\ell+1}$ which moreover is irreducible (see Vilenkin and Klymik [13], §9.2.6 e.g.) The matrices $D^\ell(g)$ are the so-called Wigner's D-matrices whose entries are, in terms of the Euler angles

$$D^\ell_{mm'}(\alpha, \beta, \gamma) = e^{-im\alpha}\, d^\ell_{mm'}(\beta)\, e^{-im'\gamma},$$

where

$$
\begin{aligned}
d^\ell_{mm'}(\beta) \;=\;& (-1)^{\ell-m'}\left[(\ell+m)!\,(\ell-m)!\,(\ell+m')!\,(\ell-m')!\right]^{1/2} \\
&\times \sum_{k=0}^{\max(\ell-m,\ell-m')} \frac{(\cos\frac{\beta}{2})^{2k-m-m'}(\sin\frac{\beta}{2})^{2\ell-2k-m-m'}}{k!\,(\ell-m-k)!\,(\ell-m'-k)!\,(m+m'+k)!} .
\end{aligned}
$$

Note that $d^\ell_{mm'}(0) = \delta^{m'}_m$, where $\delta^{m'}_m$ denotes the Kronecker delta function. It is immediate that, the Lebesgue measure of S^2 being invariant by the action of $SO(3)$,

$$
\begin{aligned}
a_{\ell.} &= \int_{S^2} T(x)Y_{\ell.}(x)\,dx \overset{d}{=} \int_{S^2} T(gx)Y_{\ell.}(x)\,dx \\
&= \int_{S^2} T(x)D^\ell(g^{-1})Y_{\ell.}(x)\,dx = D^\ell(g^{-1})a_{\ell.} ,
\end{aligned}
\tag{5}
$$

In particular, in coordinates,

$$a_{\ell m} \overset{d}{=} \sum_{m'=-\ell}^{\ell} a_{\ell m'} D^\ell_{m'm}(g) . \tag{6}$$

It is well known that for an isotropic field which is continuous in mean square the development (1) holds, the convergence being in L^2 (see [8] e.g.).

It is also useful to point out that, because of (3), the coefficients satisfy the following identity

$$a_{\ell m} = (-1)^m \overline{a_{\ell,-m}}.$$

In particular $a_{\ell 0}$ is real.

3 General properties of the spherical harmonics coefficients

It is well-known that, for mean square continuous and isotropic random fields, the spherical harmonics coefficients are orthogonal, i.e.

$$E[a_{\ell_i m_i} \overline{a_{\ell_j m_j}}] = \delta_{\ell_i}^{\ell_j} \delta_{m_j}^{m_i} C_{\ell_i} , \tag{7}$$

the sequence $\{C_\ell\}_{\ell=1,2,\dots}$ denoting the angular power spectrum of these fields. For reasons of completeness we give now a proof of this fact. Actually our statement is slightly stronger. As usual we denote by A^* the complex conjugate of the matrix A.

Proposition 1 *Assume T isotropic. Then*
    a) for all ℓ such that $E[|a_\ell.|^2] < \infty$,

$$Ea_{\ell.}a_{\ell.}^* = C_\ell I_{2\ell+1} ,$$

where $I_{2\ell+1}$ denotes the $(2\ell+1) \times (2\ell+1)$ identity matrix
    b) for all ℓ_1, ℓ_2 such that $E[|a_{\ell_1.}|^2] < \infty$, $E[|a_{\ell_1.}|^2] < \infty$

$$Ea_{\ell_1.}a_{\ell_2.}^* = 0$$

(in the sense of the $(2\ell_1+1) \times (2\ell_2+1)$ zero matrix).

Proof a) Let us denote by Γ_ℓ the covariance matrix of the random vector $a_{\ell.}$. Since the vectors $a_{\ell.}$ and $D^\ell(g)a_{\ell.}$ have the same distribution, they have the same covariance matrix. This gives

$$\Gamma_\ell = D^\ell(g)\Gamma_\ell D^\ell(g)^* = D^\ell(g)\Gamma_\ell D^\ell(g)^{-1}$$

Since D^ℓ is an irreducible representation of $SO(3)$, by Schur lemma Γ_ℓ is of the form $C_\ell I_{2\ell+1}$.
    b) The representations D^{ℓ_1} and D^{ℓ_2} are not equivalent for $\ell_1 \neq \ell_2$, having different dimensions. Therefore again by Schur lemma, the identity

$$Ea_{\ell_1.}a_{\ell_2.}^* = D^{\ell_1}(g)Ea_{\ell_1.}a_{\ell_2.}^* D^{\ell_1}(g)^{-1}$$

can hold only if the right hand side is the zero matrix.

5

Remark 2 We stress that (for strongly isotropic fields) Proposition 1 is strictly stronger than the standard result on mean square random fields. Indeed, it is immediate to show that $ET^2 < \infty$ implies $\sum_{l=1}^{\infty} E|a_{l.}|^2 < \infty$, on the other hand, it is not difficult to find examples where mean square continuity fails but the assumptions of Proposition 1 are fulfilled. Consider for instance the field:

$$T(x) = \sum_{m=-\ell_1}^{\ell_1} a_{\ell_1 m_1} Y_{\ell_1 m_1}(x) + \sum_{m=-\ell_2}^{\ell_2} a_{\ell_2 m_2} Y_{\ell_2 m_2}(x) + \sum_{m=-\ell_3}^{\ell_3} b_{\ell_3 m_3} Y_{\ell_3 m_3}(x) \quad (8)$$

where $b_{\ell_3 m_3} = \eta a_{\ell_3 m_3}$; we assume that the $a_{\ell_i m_i}$'s $(i = 1, 2, 3)$ satisfy (7) whereas η is a random variable with infinite variance (for instance a Cauchy). It is not difficult to see that the field T is properly defined and strictly isotropic; although (8) is clearly an artificial model, some closely related field may be of interest for practical applications: for instance in CMB data analysis it is often the case that the observed field is a superposition of signal plus foreground contamination, and the latter may be characterized by heavy tails at the highest multipoles (point sources). In such cases, it is of an obvious statistical interest to know that the standard properties of the spherical harmonics coefficients still hold at least for the multipoles where foreground contamination is absent. It is immediate to see that $ET^2 = \infty$, whence the field cannot be mean-square continuous; however (2) is still properly defined for $l = l_1, l_2$ (simply exchange the integral with the finite sum), and therefore Proposition 1 holds for these two vectors of spherical harmonics coefficients.

Our next proposition provides three further characterizations for the spherical harmonics coefficients of isotropic fields.

Proposition 3 *Let T be an isotropic random field (not necessarily mean square continuous); then for all $a_{l.}$ such that $E|a_{\ell.}|^2 < \infty$,*
   a) for all $m = 1, \ldots, \ell$,

$$\operatorname{Re} a_{\ell m} \stackrel{d}{=} \operatorname{Im} a_{\ell m} \quad and \quad \frac{\operatorname{Re} a_{\ell m}}{\operatorname{Im} a_{\ell m}} \sim Cauchy$$

   b) for all $m = 1, 2, \ldots, \ell$, $\operatorname{Re} a_{\ell m}$ and $\operatorname{Im} a_{\ell m}$ are uncorrelated, with variance $E(\operatorname{Re} a_{\ell m})^2 = E(\operatorname{Im} a_{\ell m})^2 = C_\ell/2$.
   c) The marginal distribution of $\operatorname{Re} a_{\ell m}$, $\operatorname{Im} a_{\ell m}$ is always symmetric, that is,

$$\operatorname{Re} a_{\ell m} \stackrel{d}{=} -\operatorname{Re} a_{\ell m} , \quad \operatorname{Im} a_{\ell m} \stackrel{d}{=} -\operatorname{Im} a_{\ell m} .$$

Proof a) For $\beta = \gamma = 0$, (6) becomes

$$a_{\ell m} \stackrel{d}{=} e^{-im\alpha} a_{\ell m}$$

for all $m = -\ell, \ldots, \ell, 0 \le \alpha < 2\pi$. This entails

$$\begin{pmatrix} \operatorname{Re} a_{\ell m} \\ \operatorname{Im} a_{\ell m} \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \operatorname{Re} a_{\ell m} \\ \operatorname{Im} a_{\ell m} \end{pmatrix} \tag{9}$$

for all $m = -\ell, \ldots, \ell, 0 \le \varphi < 2\pi$. Thus the vector ${}^t(\operatorname{Re} a_{\ell m}, \operatorname{Im} a_{\ell m})$ has a distribution that is invariant by rotations, that is, in polar coordinates, they can be written in of the form

$$R\cos(\Theta) \tag{10}$$

where R is a random variable with values in $\mathbb{R}^+$, whereas Θ is uniform in $[-\pi, \pi]$. This entails immediately that $\arctan(\operatorname{Re} a_{\ell m} / \operatorname{Im} a_{\ell m}) \sim U(-\frac{\pi}{2}, \frac{\pi}{2})$; the result follows immediately.

b) This property is well-known if T is mean square continuous. From (10),

$$E[\operatorname{Re} a_{\ell m} \cdot \operatorname{Im} a_{\ell m}] = \int_0^{+\infty} r^2 d\mu_R(r) \int_{-\pi}^{\pi} \cos\vartheta \sin\vartheta d\vartheta = 0 \ .$$

c) It suffices to take $\varphi_0 = \pi$ in (9).

Remark 4 It is interesting to note how Proposition 1 implies that no information can be derived on the statistical distribution of an isotropic random field by the marginal distribution function of the ratios $(\operatorname{Re} a_{\ell m} / \operatorname{Im} a_{\ell m})$. On the other hand, it may be possible to use these ratios to implement statistical tests of the assumption of isotropy, an issue which has gained a remarkable empirical relevance after the first release of the WMAP data in February 2003

It is clear that if the field T is gaussian, then the r.v.'s $(a_{\ell m})_{\ell,m}$ is a gaussian family. We prove now an independence result for this family of r.v.'s. Thanks to Proposition 1, these r.v.'s are uncorrelated, but one must be careful, since, in the case of complex r.v.'s, absence of correlation and joint gaussian distribution does not imply independence.

Proposition 5 *For an isotropic gaussian random field the r.v.'s $a_{\ell m}$, $\ell = 0, 1, \ldots, m = 0, \ldots, \ell$ are independent.*

Proof Let be $(\ell, m) \neq (\ell', m')$, $m > 0$, $m' > 0$. Then $a_{\ell m}$ is uncorrelated with both $a_{\ell' m'}$ and $a_{\ell', -m'} = \overline{a_{\ell' m'}}$. Thus

$$E[a_{\ell m} \overline{a_{\ell' m'}}] = 0, \qquad E[a_{\ell m} a_{\ell' m'}] = E[a_{\ell m} \overline{a_{\ell', -m'}}] = 0$$

and the statement follows from Lemma 6. If one at least among m and m' is equal to 0, then the r.v. $a_{\ell m}$ (or $a_{\ell' m'}$ is real and independence follows from absence of correlation as for the real case.

Lemma 6 *Let Z_1, Z_2 be complex r.v.'s, centered and jointly gaussian. Then they are independent if and only if*

$$E[Z_1 \overline{Z}_2] = 0, \qquad E[Z_1 Z_2] = 0 \tag{11}$$

Proof In one direction the statement is obvious. Let us assume that (11) are satisfied. Then, if we set $Z_k = X_k + iY_k$, $k = 1, 2$, then

$$E[X_1 X_2 + Y_1 Y_2] + iE[-X_1 Y_2 + Y_1 X_2] = 0$$
$$E[X_1 X_2 - Y_1 Y_2] + iE[X_1 Y_2 + Y_1 X_2] = 0$$

From these one obtains $E[X_1 X_2] = 0$, $E[Y_1 Y_2] = 0$, $E[X_1 Y_2] = 0$ and $E[Y_1 X_2] = 0$. This means that each of the r.v.'s X_1, Y_1, X_2, Y_2 is uncorrelated with the other ones, so that, being jointly gaussian, they are independent.

Which is less obvious, is that the converse also holds. The following is the main result of this paper.

Theorem 7 *For an isotropic random field, let ℓ be such that $E|a_\ell|^2 < \infty$. Then the coefficients $(a_{\ell 0}, a_{\ell 1} \ldots, a_{\ell \ell})$ are independent if and only if they are gaussian.*

Proof We just need to prove the "only if" part. Fix $m_1 \geq 0$, $m_2 \geq 0$, so that the two complex r.v.'s $a_{\ell m_1}$ and $a_{\ell m_2}$ are independent. Note that we are not assuming the independence of $\mathrm{Re}\, a_{\ell m_1}$ and $\mathrm{Im}\, a_{\ell m_1}$ or of $\mathrm{Re}\, a_{\ell m_2}$ and $\mathrm{Im}\, a_{\ell m_2}$. Thanks to (5), the two vectors a_ℓ and $D^\ell(g)a_\ell$ have the same distribution. Thus the two r.v.'s

$$L_1 = \sum_{m'=-\ell}^{\ell} D^\ell_{m', m_1}(g) a_{\ell m'} \quad \text{and} \quad L_2 = \sum_{m'=-\ell}^{\ell} D^\ell_{m', m_2}(g) a_{\ell m'}$$

8

having the same joint distribution as $a_{\ell m_1}$ and $a_{\ell m_2}$, are independent. Fix g so that the angles α such that $m\alpha \neq k\pi$ for all integers m, k and β such that $d^\ell_{mm_1}(\beta)$ and $d^\ell_{mm_2}(\beta)$ are different from zero for all $m = -\ell, \ldots, \ell$ (note that such a β certainly exists, because the functions $d^\ell_{m_2 m}(\beta)$, $m, m' = -\ell, \ldots, \ell$ are analytic and can vanish only at a finite number of values $\beta \in [0, \pi]$). For such a choice of g, thanks to (5),

$$\begin{pmatrix} \operatorname{Re} L_1 \\ \operatorname{Im} L_1 \end{pmatrix} = \sum_{m=-\ell}^{\ell} d^\ell_{mm_1}(\beta) \begin{pmatrix} \cos m_1\alpha & \sin m_1\alpha \\ -\sin m_1\alpha & \cos m_1\alpha \end{pmatrix} \begin{pmatrix} \operatorname{Re} a_{\ell m} \\ \operatorname{Im} a_{\ell m} \end{pmatrix}$$

and

$$\begin{pmatrix} \operatorname{Re} L_2 \\ \operatorname{Im} L_1 \end{pmatrix} = \sum_{m=-\ell}^{\ell} d^\ell_{mm_2}(\beta) \begin{pmatrix} \cos m_2\alpha & \sin m_2\alpha \\ -\sin m_2\alpha & \cos m_2\alpha \end{pmatrix} \begin{pmatrix} \operatorname{Re} a_{\ell m} \\ \operatorname{Im} a_{\ell m} \end{pmatrix}$$

where the 2×2 matrices on the right hand-sides are always full rank. By the Skitovich-Darmois theorem below (see Kagan, Rao and Linnik [7] e.g.), it follows that each of the vectors $(\operatorname{Re} a_{\ell m}, \operatorname{Im} a_{\ell m})$ is bivariate Gaussian; as $(\operatorname{Re} a_{\ell m}, \operatorname{Im} a_{\ell m})$ are uncorrelated and have the same variance by Proposition 1, then $a_{\ell m} = \operatorname{Re} a_{\ell m} + i \operatorname{Im} a_{\ell m}$ is complex Gaussian.

Theorem 8 *(Skitovich-Darmois) Let $X_1, \ldots, X_r$ be mutually independent random vectors in R^n. If the linear statistics*

$$L_1 = \sum_{j=1}^{r} A_j X_j, \qquad L_2 = \sum_{j=1}^{r} B_j X_j \ ,$$

are independent, for some real nonsingular $n \times n$ matrices A_j, B_j, $j = 1, \ldots, r$, then each of the vectors $X_1, \ldots, X_r$ is normally distributed.

In particular Theorem 7 implies that if an isotropic random field is mean square continuous and the coefficients $a_{\ell m}, \ell = 0, 1, \ldots, m = 0, \ldots \ell$ are independent, then it is gaussian.

Remark 9 Proposition 7 shows that it is not possible to generate isotropic random fields by sampling non-Gaussian, independent complex-valued random variables $a_{\ell m}, m = -\ell, \ldots, \ell$. This fact shows that, apart from the gaussian case, it is not easy to sample a random field by simulating the values of the random coefficients $a_{\ell m}$. In particular sampling independent values of the $a_{\ell m}$'s

with distributions other than the gaussian gives not rise to an isotropic random field.

We wish also to point out that it is indeed possible to construct a non-gaussian random field by choosing the random coefficients $a_{\ell m}$, $\ell = 0, 1, \ldots$ $m = 0, \ldots, \ell$ independent and with an arbitrary distribution. If they satisfy the conditions of Proposition 1 and the series (1) converges, they certainly define a random field on S^2. But Theorem 7 states that such a field cannot be isotropic. In particular Theorem 7 does not follow by any means from the central limit theorem.

References

[1] M. Arjunwadkar, C. R. Genovese, C. J. Miller, R. C. Nichol, and L. Wasserman, *Nonparametric inference for the cosmic microwave background.*, Statist. Sci., 19 (2004), pp. 308–321.

[2] N. Bartolo, E. Komatsu, S. Matarrese, and A. Riotto, *Non-Gaussianity from inflation: theory and observations*, Phys. Rep., 402 (2004), pp. 103–266.

[3] C. R. Contaldi and J. Magueijo, *Generating non-gaussian maps with a given power spectrum and bispectrum*, Phys. Rev. D, 63 (2001).

[4] M. Cruz, E. Marinez-Gonzalez, P. Vielva, and L. Cayon, *Detection of a non-gaussian spot in WMAP*, Monthly Notices of R. Astronomical Society, 356 (2005), pp. 29–40.

[5] O. Doré, S. Colombi, and F. Bouchet, *Probing CMB non Gaussianity using local curvature*, Monthly Notices of R. Astronomical Society, 344 (2003), pp. 905–916.

[6] J. Jin, J.-L. Starck, D. Donoho, N. Aghanim, and O. Forni, *Cosmological non-gaussian signature detection: Comparing performance of different statistical tests.*, Eurasip J. Appl. Signal Processing, (2005), p. forthcoming.

[7] A. M. KAGAN, Y. V. LINNIK, AND C. R. RAO, *Characterization problems in mathematical statistics*, John Wiley & Sons, New York-London-Sydney, 1973. Translated from the Russian by B. Ramachandran, Wiley Series in Probability and Mathematical Statistics.

[8] N. LEONENKO, *Limit theorems for random fields with singular spectrum*, vol. 465 of Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 1999.

[9] D. MARINUCCI, *Testing for non-gaussianity on cosmic microwave background radiation: A review*, Statist. Sci., 19 (2004), p. 294307.

[10] D. MARINUCCI AND M. PICCIONI, *The empirical process on Gaussian spherical harmonics*, Ann. Statist., 32 (2004), pp. 1261–1288.

[11] C. PARK, *Non gaussian signatures in the temperature fluctuations observed by the Wilkinson Microwave Anisotropy Probe*, Monthly Notices of R. Astronomical Society, 349 (2004), pp. 313–320.

[12] D. A. VARSHALOVICH, A. N. MOSKALEV, AND V. K. KHERSONSKIĬ, *Quantum theory of angular momentum*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1988. Irreducible tensors, spherical harmonics, vector coupling coefficients, $3nj$ symbols, Translated from the Russian.

[13] N. J. VILENKIN AND A. U. KLIMYK, *Representation of Lie groups and special functions. Vol. 2*, vol. 74 of Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, 1993. Class I representations, special functions, and integral transforms, Translated from the Russian by V. A. Groza and A. A. Groza.